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LETTERS TO THE EDITOR

PERIODIC SOLUTIONS AND HARMONIC ANALYSIS OF AN ANTI-LOCK BRAKE SYSTEM WITH PIECEWISE-NONLINEARITY

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1. INTRODUCTION

An anti-lock brake system (ABS) is used to prevent lock-up of an automobile wheel at the moment of an emergency brake so that good directional stability and short stopping distance are ensured. The simplest dynamic model for ABS, i.e., the single-wheel ABS, is a two-dimensional non-linear dynamical system with switching control, where the relationship between the longitudinal friction coefficient and slip is the most essential non-linear factor. Under the action of the different control laws, the ABS dynamics may show various forms of closed orbits (limit cycles). Fling and Feston [1] presented an anti-skid or anti-lock design approach on the basis of describing-function theory, in which a non-linear compensatory is used to fix the amplitude and frequency of the desired oscillation (stable small-amplitude limit cycle) in an anti-skid mode. Yeh [2] introduced the Poincaré map concept into ABS dynamics to investigate the effects of various anti-lock control laws on the limit cycle using numerical approach. Kuo [3] studied a four-phase control scheme of ABS using the general expressions of the limit cycle in phase plane. Cheng [4] investigated a three-phase control law of ABS using Yeh's approach [2].

In this work, a semi-analytical and semi-numerical approach is proposed to analyze the periodic solutions and stability of the single-wheel ABS. The same assumptions as Kuo's [3] are considered, which are the piecewise linear tyre model with two segments and the slow variation of vehicle velocity. On the basis of these, the piecewise expressions for the periodic solution in every interval are derived analytically, while the lengths of every interval and the period of solution, etc. are found numerically, which are provided in symbol algebra software MAPLE 4.0. The results obtained by the present approach are compared to the numerical one. Taking account of the requirement for the analysis of coupling vibrations, this work has also studied the harmonic solutions of the ABS.

2. NON-LINEAR DYNAMICAL MODEL FOR ABS

A practical ABS model is a complicated non-linear dynamical system including a number of non-linear factors. In this paper, the single-wheel ABS model of Yeh [3] is used, which has been adopted by many researchers, i.e.,

$$M\dot{V} = -F_x, \tag{1}$$

$$I\dot{\omega} = -(T_b - rF_x), \tag{2}$$

$$\dot{T}_b = U, \tag{3}$$

where M is the total mass of vehicle, V the vehicle velocity, F_x the tyre force between wheel and road, I the moment of inertia of wheel, ω the angular velocity of wheel, r the rolling radius of wheel, T_b the brake torque applied to wheel and U the brake torque change rate.

The longitudinal friction F_x can be expressed as

$$F_x = \mu N = \mu Mg, \tag{4}$$

where N is the normal force applied to wheel, g the gravitational acceleration, and μ the longitudinal friction coefficient. A lot of experiments indicate that there exists a non-linear relationship between μ and the wheel slip S, which can be described by a continuous non-linear function, a piecewise non-linear or a piecewise linear function.

The slip S is defined as

$$S = 1 - \frac{r\omega}{V}. \tag{5}$$

In this study, the non-linear relationship between μ and S is approximately expressed as

$$\mu = \begin{cases} k_A S, & 0 \leqslant S \leqslant S_c, \\ \mu^* + k_B S, & S_c < S \leqslant 1, \end{cases} \tag{6}$$

where $k_A = \mu_c/S_c$, $k_B = (\mu_g - \mu_c)/(1 - S_c)$, $\mu^* = (\mu_c - \mu_g S_c)/(1 - S_c)$, μ_c, μ_g and S_c are constants. For $S \in [0, S_c]$, $k_A > 0$, but for $S \in [S_c, 1]$, $k_B < 0$. μ reaches the peak value μ_c when $S = S_c$.

In equation (3), generally, the change rate U may have three different constants, i.e.,

In the pressure increasing mode: $U = U_i > 0$, when $T_b \geqslant rF_x - H_3$,

In the pressure decreasing mode: $U = U_d < 0$, when $T_b > rF_x + H_1$, (7)

In the pressure holding mode: $U = 0$, when $T_b < rF_x - H_3$,

where H_1 and H_3 are the threshold values of prediction boundary P1 and reselection boundary R3 [3] respectively.

In an ABS braking, generally, the variation of vehicle velocity V is much slower than that of S, ω and T_b so that V can be considered as a constant when the periodic solutions of S, ω and T_b are studied. In this case, equations (1)–(3) can be simplified as usual ABS dynamics equations [3]:

$$\dot{S} = \frac{r[T_b(t) - T_e(S)]}{IV}, \tag{8}$$

$$\dot{T}_b = U, \tag{9}$$

where $T_e(S)$ is called the equilibrium torque, which represents the equilibrium solution curve of ABS dynamics equations (8)–(9), and is defined as

$$T_e(S) = rF_x\left[1 + \frac{(1-S)I}{r^2 M}\right] = r\mu N\left[1 + \frac{(1-S)I}{r^2 M}\right]. \tag{10}$$

From equations (4), (6), (7), and (10), one can see that equations (8), (9) represent a two-dimensional, piecewise-non-linear autonomous system. In equation (8), V is referred to as a constant, while the effect of slow variation of V on the ABS dynamics will be discussed elsewhere.

3. PERIODIC SOLUTIONS OF SINGLE-WHEEL ABS

As stated above, the control action of an ABS is to hold the friction coefficient μ near its peak value μ_c in braking to make the friction force F_x reach a bigger value, and to keep the wheel from being locked up. Under the control of the boundaries P1 and R3, an ABS may form a closed trajectory $1 \to 2 \to 3 \to 4 \to 5 \to 1$ in the S–T_b phase plane, i.e., a limit cycle, as shown in Figure 1, which can be obtained by numerically integrating equations (8), (9).

For periodic solutions of some low-dimensional, piecewise-linear dynamical systems [5–7], a semi-analytical and semi-numerical approach seems to be more accurate, rapid and effective than direct numerical integration, in which the solution in each time interval is analytically expressed, while the unknowns are numerically solved from non-linear algebraic equations. For the piecewise-non-linear system (8), (9) in this paper, however, it is impossible to obtain an accurate analytical solution in each interval. In this study, on the basis of an approximate assumption, a semi-analytical and semi-numerical method can be used to find the periodic solutions of the ABS.

3.1. FORM OF LIMIT CYCLE

From Figure 1 it is known that the limit cycle is a clockwise cycling closed trajectory, which can be regarded as one consisting of the following five time intervals: $T_{1\to 2} = \{t \in [t_1, t_2] | U = U_i > 0, \quad k_A > 0\}$, $T_{2\to 3} = \{t \in [t_2, t_3] | U = U_i > 0, \quad k_B < 0\}$, $T_{3\to 4} = \{t \in [t_3, t_4] | U = U_d < 0, \quad k_B < 0\}$, $T_{4\to 5} = \{t \in [t_4, t_5] | U = 0, \quad k_B < 0\}$ and $T_{5\to 1} = \{t \in [t_5, T] | U = 0, k_A > 0\}$. t_1 corresponds to a start point at the limit cycle, and T is the cycle length around the limit cycle, i.e., the period of a steady solution of the ABS. Letting $\tau_j = t - t_j$, $T_j = t_{j+1} - t_j$, $j = 1, 2, 3, 4, 5$, one has $0 \leqslant \tau_j \leqslant T_j$.

3.2. APPROXIMATE ANALYTICAL SOLUTIONS OF ABS DYNAMICAL EQUATIONS

In equation (8), letting

$$x(t) = T_b(t) - T_e[S(t)] \tag{11}$$

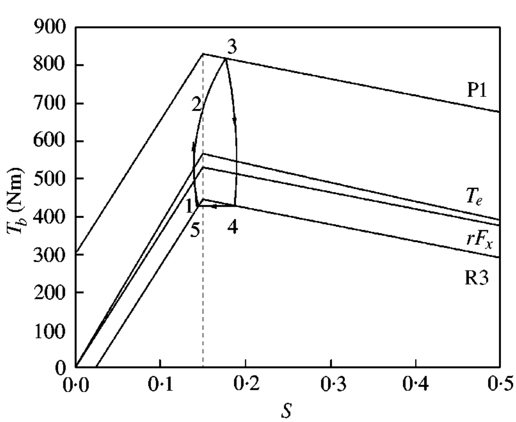

Figure 1. The limit cycle in S–T_b phase plane.

and differentiating equation (11) with respect to time, one has

$$\dot{x} = \dot{T}_b - T'_e\dot{S} = \dot{T}_b - rT'_e x/IV, \tag{12}$$

where the dot denotes $(\mathrm{d}/\mathrm{d}t)(\cdot)$, and the prime denotes $(\mathrm{d}/\mathrm{d}S)(\cdot)$. From equations (6) and (10) it is found that $T_e(S)$ is a square function of S.

As the value of I/r^2M is very small, $T_e(S)$ approaches quite closely rF_x, i.e., $T_e(S) \approx rF_x = r\,\mu N$. In equation (12), hence, the approximate assumption $T'_e(S) = rkN$ can be done, where $k = k_A$, or $k = k_B$. Integrating equations (9) and (12), and taking account of $\tau_j = t - t_j$, one can obtain the approximate analytical solution of equations (8) and (9) in the jth interval:

$$S_j(\tau_j) = S_j(0) + \frac{1}{b_j}\left[\frac{U_j}{\alpha_j} - x_j(0)\right](1 - \mathrm{e}^{-\alpha_j\tau_j}) - \frac{a_j}{b_j^2}\ln\left\{\frac{1 + \dfrac{b_j}{a_j}\left[\dfrac{U_j}{\alpha_j} - \left(\dfrac{U_j}{\alpha_j} - x_j(0)\right)\mathrm{e}^{-\alpha_j\tau_j}\right]}{1 + \dfrac{b_j}{a_j}x_j(0)}\right\}, \tag{13}$$

$$T_{bj}(\tau_j) = U_j\tau_j + T_{bj}(0) \tag{14}$$

where $x_j(0) = T_{bj}(0) - T_{ej}[S_j(0)] = T_{bj}(0) - c_jS_j(0) - d_jS_j^2(0) - \beta_j$, $a_j = IVU_j/r$, $b_j = -rk_jN$, $\alpha_j = r^2k_jN/IV$, $U_j = U_i$, U_d or 0 and $k_j = k_A$ or k_B. $c_j = k_jN(r + I/Mr)$, $d_j = -\,k_jIN/Mr$, $\beta_j = 0$, while $j = 1, 5$; $c_j = k_jN(r + I/Mr) - \mu^*IN/Mr$, $d_j = -\,k_jIN/Mr$, $\beta_j = \mu^*N(r + I/Mr)$, while $j = 2, 3, 4$.

3.3. TIME INTERVAL LENGTH T_j AND PERIOD T

To determine every time interval length T_j and the period of solution T, it is necessary to use the following continuity and periodicity conditions:

$$\begin{aligned}
&S_1(T_1) = S_2(0) = S_c, \quad T_{b1}(T_1) = T_{b2}(0),\\
&S_2(T_2) = S_3(0), \quad T_{b2}(T_2) = T_{b3}(0) = rN[\mu^* + k_2S_2(T_2)] + H_1,\\
&S_3(T_3) = S_4(0), \quad T_{b3}(T_3) = T_{b4}(0) = rN[\mu^* + k_3S_3(T_3)] - H_3,\\
&S_4(T_4) = S_5(0) = S_c, \quad T_{b4}(T_4) = T_{b5}(0),\\
&T_{b1}(0) = -\,b_1S_1(0) - H_3, \quad T_{b5}(T_5) = -\,b_5S_5(T_5) - H_3,\\
&S_5(T_5) = S_1(0), \quad T_{b5}(T_5) = T_{b1}(0),
\end{aligned} \tag{15}$$

where $S_1(0)$ and $T_{b1}(0)$ are initial conditions. From equations (13)–(15), one can obtain the equations for T_1, T_2 and T_3, and in this work T_1, T_2 and T_3 are found by means of MAPLE. T_4 and T_5 must be solved from the equation

$$\tau_j = -\frac{1}{\alpha_j}\ln\left\{1 + \frac{b_j[S_j(\tau_j) - S_j(0)]}{x_j(0)}\right\}, \quad j = 4, 5 \tag{16}$$

instead of from equations (13) and (14). Equation (16) can be deduced from $\mathrm{d}x/\mathrm{d}S = -\,T'_e$ or from equation (13) by letting the third term be zero.

From the above solving process one obtains that the solutions of the system are related to the initial value of the point $[S_1(0),\ T_{b1}(0)]$. Hence, an iterative process is needed to converge to the final limit cycle. In this way, matching the solutions in the intervals forms the periodic solution, and the period of the solution is given by

$$T = \sum_{j=1}^{5} T_j. \tag{17}$$

In the next section, it is indicated that the periodic solution and the period can be found by solving the fixed point of the Poincaré map.

4. STABILITY OF PERIODIC SOLUTIONS

The stability analysis of the periodic solution can be carried out by constructing the Poincaré map [8]. In this work, constructing the Poincaré map is to find such a function relationship: $S_5(T_5) = P[S_1(0)]$. From equations (14) and (15), with respect to $U_4 = U_5 = 0$, one can obtain

$$S_5(T_5) = P[S_1(0)] = S_1(0) - \{U_3T_3[S_1(0)] + U_2T_2[S_1(0)] + U_1T_1[S_1(0)]\}/b_5, \tag{18}$$

where the expressions of $T_1[S_1(0)]$, $T_2[S_1(0)]$ and $T_3[S_1(0)]$ can be solved by using the "solve" function of MAPLE V. Equation (18) is just the Poincaré map to construct.

To examine the stability of the periodic solution, it is necessary to find a fixed point of the Poincaré map, i.e., the solution $S_1^*(0)$ satisfying

$$S_5(T_5) - S_1(0) = 0, \tag{19}$$

which can be found by using the "fsolve" function of MAPLE V. Once $S_1^*(0)$ is obtained, every time interval $T_j[S_1^*(0)]$ $j = 1$–5 and the period T are determined. Hence, the periodic solution of the system can also be found by directly solving the fixed point of the Poincaré map. The iteration solving the periodic solution in the above section is, in fact, just the process that the Poincaré map converges to the fixed point.

According to the derivative of the Poincaré map $P[S_1(0)]$ at the fixed point $S_1^*(0)$

$$P'[S_1^*(0)] = \left.\frac{dS_5(T_5)}{dS_1(0)}\right|_{S_1(0)=S_1^*(0)}, \tag{20}$$

the stability of the periodic solution can be examined. If $P'[S_1^*(0)] < 1$, then the periodic solution is stable and if $P'[S_1^*(0)] > 1$, then the periodic solution is unstable. The derivative $P'[S_1^*(0)]$ can be obtained by using the "diff" function of MAPLE V.

5. COMPUTATIONAL RESULTS

In this study, the parameters in reference [3] are used: $I = 2{\cdot}16\ \text{kgm}^2$, $M = 300$ kg, $r = 0{\cdot}3$ m, $U_i = 5292$ Nm/s, $U_d = -15876$ Nm/s, $V = 25$ m/s, $H_1 = 300$ Nm, $H_3 = 85$ Nm, $S_c = 0{\cdot}15$, $k_A = 4$, $k_B = -0{\cdot}5$.

Finding the periodic solution by the iterative method is first carried out. Letting $S_1(0) = 0{\cdot}1$, one has the first iterative result: $S_5(T_5) = 0{\cdot}1469456265 \neq S_1(0)$. After eight iterations, we obtain $S_5(T_5) = S_1^*(0) = 0{\cdot}1453521866$, i.e., the solution converges to the fixed point. The corresponding time intervals are $T_1 = 0{\cdot}04808474439$, $T_2 = 0{\cdot}0255226458$, $T_3 = 0{\cdot}02453579673$, $T_4 = 0{\cdot}05296374200$, $T_5 = 0{\cdot}006484782469$, respectively, and the

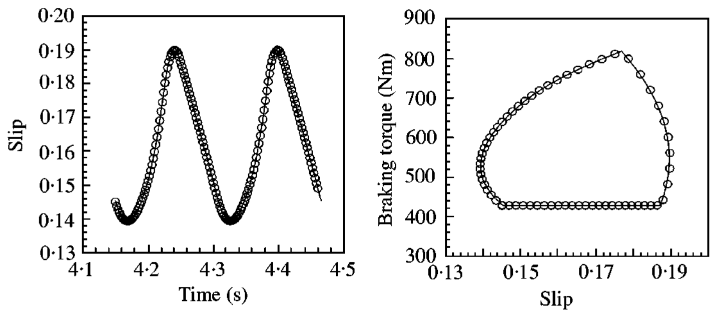

Figure 2. Comparisons of approximate analytical results (——) and numerical results (○).

period $T = 0{\cdot}1575917115$. As a result, we can obtain the approximate periodic solution from equations (13) and (14) by the approach in this paper. The comparisons between the approximate analytical results and the numerical results obtained by the improved Euler method are given in Figure 2.

It is worth noting that the results of direct numerical integration are related to the time step, i.e., the smaller the step, the closer to the approximate analytical results are the numerical results. Obviously, the time step must be very small by direct numerical integration to approach the accuracy of the approximate analytical results. As a result, it must take quite a long time.

Using the method of finding the fixed point of the Poincaré map, one has $S_5(T_5) = S_1^*(0) = 0{\cdot}1453521862$, $T = 0{\cdot}1575917167$, which agree with the results obtained by the iterative method.

The derivative obtained by computing $P'[S_1^*(0)] = 0{\cdot}062756277 < 1$ shows that the periodic solution of the system is stable.

6. MULTI-HARMONIC ANALYSIS OF ABS

Apparently, the periodic solutions of the ABS obtained in this paper are periodic functions that may include several harmonics. Analyzing the composition and amplitude distribution of the harmonics is important to study coupling vibrations between low and high frequencies, and mechanic–hydraulic coupling vibrations in complicated ABS models, as well as torsional vibrations in vehicle transmission systems.

The ABS model in this paper, as above, is a two-dimensional, piecewise-non-linear autonomous system. The multi-harmonic analysis of the periodic solutions of the ABS can be carried out using the simple and accurate HB-FFT method proposed by Choi and Lou [9], which combines the harmonic balance method and fast Fourier transformation (FFT) procedure.

The periodic solutions of the ABS can be represented by the Fourier series:

$$S = S_0 + \sum_{n=1}^{N_0} (a_n \cos n\,\omega t + b_n \sin n\,\omega t), \tag{21}$$

$$T_b = T_{b0} + \sum_{n=1}^{N_0} (c_n \cos n\,\omega t + d_n \sin n\,\omega t), \tag{22}$$

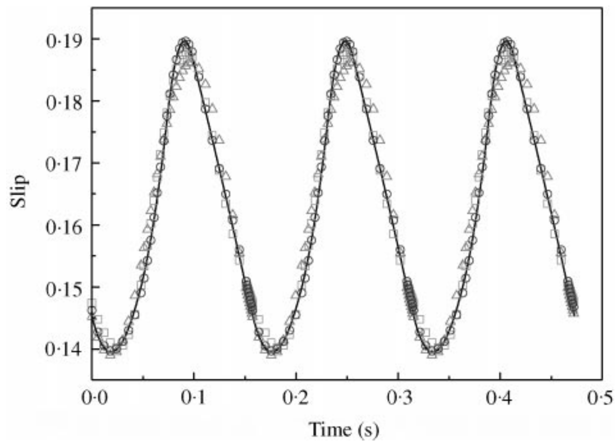

Figure 3. Comparison of harmonic solutions (△, $N_0 = 1$; □, $N_0 = 2$; ○, $N_0 = 3$) and approximate analytical solution (——).

where N_0 is the number of harmonics to be taken into account, $\omega = 2\pi/T$. Similarly, the piecewise-non-linear equilibrium torque

$$T_e(S) = \begin{cases} l_{A1}S + l_{A2}S^2 & S \leqslant S_c, \\ l_{B0} + l_{B1}S + l_{B2}S^2 & S > S_c, \end{cases} \tag{23}$$

where $l_{A1} = k_A N(r + I/Mr)$, $l_{A2} = -k_A NI/Mr$, $l_{B0} = \mu^* N(r + I/Mr)$ *and* $l_{B1} = k_B N(r + I/Mr) - \mu^* NI/Mr$, $l_{B2} = -k_B NI/Mr$, can also be expressed as a Fourier series of the following form:

$$T_e(S) = T_{e0} + \sum_{n=1}^{N_0} (\alpha_n \cos n\,\omega t + \beta_n \sin n\,\omega t). \tag{24}$$

The Fourier coefficients in equation (22) T_{b0}, c_n and d_n can be analytically calculated in terms of

$$T_b(t) = \begin{cases} U_i + T_{b\,min}, & 0 \leqslant t < t_3, \\ -U_d(t - t_4) + T_{b\,min}, & t_3 \leqslant t < t_4, \\ T_{b\,min}, & t_4 \leqslant t < T, \end{cases} \tag{25}$$

where $t_3 = T_1 + T_2$, $t_4 = T_1 + T_2 + T_3$ and $T_{b\,min} = T_{b3}(T_3)$. The Fourier coefficients in equation (21) S_0, a_n and b_n cannot be analytically solved by integrating though $S_j(\tau_j)$ has been obtained in equation (13). Hence, the coefficients are obtained as unknowns using the harmonic balance method. From equations (21) and (23), the Fourier coefficients in equation (24) T_{e0}, α_n and β_n can be analytically expressed as the square functions of the coefficients S_0, a_n and b_n by means of MAPLE V.

Substituting equations (21), (22) and (24) into equation (8) and using the harmonic balance method will lead to the following $2N_0 + 1$ equations:

$$\begin{aligned} T_{e0}(S_0, a_n, b_n) &= T_{b0}, \\ a_n &= \frac{r}{n\omega IV}\,[\beta(S_0, a_n, b_n) - d_n], \\ b_n &= \frac{r}{n\omega IV}\,[c_n - \alpha(S_0, a_n, b_n)] \end{aligned} \tag{26}$$

for $n = 1, \ldots, N_0$. These $2N_0 + 1$ equations with $4N_0 + 2$ unknowns are non-linear because T_{e0}, α_n and β_n are each the square functions of the coefficients S_0, a_n and b_n. Once a set of initial trial values of S_0, a_n and b_n is given, the final results of the coefficients S_0, a_n and b_n will be obtained by means of the iterative algorithm [9].

The initial trial values S_{01}, a_{n1} and b_{n1} can be found from the first harmonic component without regard to the other harmonics. Using the parameters and the results given in the above section, we obtain the comparison of the time histories, i.e., the harmonic solutions from equation (21) for $N_0 = 1, 2$ and 3, respectively, and the approximate analytical solution from equation (13), as shown in Figure 3.

7. CONCLUSIONS

In this study, a semi-analytical and semi-numerical approach is presented to solve the periodic solutions of the ABS, which is a two-dimensional, piecewise-non-linear autonomous system with boundary control. This approach, for the simple system in this paper, is found to be very accurate and timesaving when compared with the direct numerical integration, in which only the use of very small time step can capture the exact switching point from one time interval to the other. Both the iterative procedure and the Poincaré map can be used to find the periodic solutions. The stability of the periodic solutions can be tested by using the derivative of the Poincaré map at the fixed point.

Using the HB-FFT method, the multi-harmonic analysis for the periodic solutions is carried out. For the given parameters in this paper, three harmonics were found to be adequate for good accuracy at a reasonable cost.

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